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ON THE EXISTENCE OF PERIODIC SOLUTIONS IN THE NONLINEAR THEORY OF OSCILLATIONS OF NONSHALLOW REISSNER SHELLS OF REVOLUTION, ACCOUNTING FOR DECAY *

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The problem of nonlinear oscillations of a homogeneous, isotropic nonshallow Reissner shell of revolution of constant thickness, with decay and periodicity of the application of the external load, is considered. The proof of existence of a generalized periodic solution and of convergence of the Bubnov—Galerkin method is given.
The problem of existence of periodic solutions of nonlinear equations of the theory of plates and shallow shells with decay, were studied in /1,2/.

1. Basic relations. We consider a peripherally closed, homogeneous isotropic nonshallow shell of revolution described by the relations

$$\varepsilon_1 = \alpha_0^{-1}(u'\cos\vartheta + w'\sin\vartheta) + \cos(\vartheta - \vartheta_0) - 1, \quad \varepsilon_2 = r_0^{-1}u$$
$$\gamma = \alpha_0^{-1}(w'\cos\vartheta - u'\sin\vartheta) - \sin(\vartheta - \vartheta_0)$$
$$\varkappa_1 = \alpha_0^{-1}(\vartheta_0' - \vartheta'), \quad \varkappa_2 = r_0^{-1}(\sin\vartheta_0 - \sin\vartheta)$$
$$T_1 = B(\varepsilon_1 + \nu\varepsilon_2), \quad M_1 = D(\varkappa_1 + \nu\varkappa_2), \quad Q = C\gamma \quad (1 \rightleftarrows 2),$$
$$B = (1-\nu^2)^{-1}Eh, \quad D = 12^{-1}(1-\nu^2)^{-1}Eh^3, \quad C = 2^{-1}(1+\nu)^{-1}Eh$$

Here a prime denotes a derivative with respect to the spatial coordinate ξ. The remaining symbols are those given in /3,4/.

The differential equations of oscillation of the shell with decay, can be written in the form

$$\mathbf{u}_{tt} + \varepsilon\mathbf{u}_t + \mathbf{A}\mathbf{u} = \mathbf{F} \tag{1.1}$$

where $\varepsilon > 0$ is a constant, F is a known vector function of time, the subscript t denotes differentiation with respect to t and A is a nonlinear operator not depending explicitly on t.

Let the shell be acted upon by time-periodic mass forces F with period ω. The problem consists of finding a vector $\mathbf{u}(\xi, t) = (u, w, \vartheta)$ $(a \leqslant \xi \leqslant b, \ -\infty < t < +\infty)$ satisfying the equations (1.1) and conditions

$$\mathbf{u}(a, t) = \mathbf{u}(b, t) = 0 \tag{1.2}$$
$$\mathbf{u}(\xi, t+\omega) = \mathbf{u}(\xi, t), \quad \mathbf{u}_t(\xi, t+\omega) = \mathbf{u}_t(\xi, t) \tag{1.3}$$

2. Basic assumptions. Let the following conditions hold:

1) the middle surface of the shell represents a surface of revolution contained between two parallel lines $\xi = a$ and $\xi = b$; the homemorphic mapping of its meridian on the segment $[a, b]$ is produced by the function $r \in C^{(2)}(a, b)$;

2) the following inequalities hold in the domain of variation of the parameters ξ $(0 < a \leqslant \xi \leqslant b < \infty)$:

$$0 < m_1 \leqslant \alpha_0^{-1}r_0, \quad E \leqslant m_2 < \infty, \quad 0 < \nu < 2^{-1}$$

where m_1 and m_2 are certain constants;

3) the units of measurement of mass density ρ and the linear dimensions of the shell are chosen such that $\rho = 1$, $h = 1$.

Basic spaces. Space $H(a, b)$ is a Hilbert space obtained by the closure of the set C_1 of vector functions $\mathbf{u} = (u, w, \vartheta) \in C^{(1)}(a, b)$ satisfying the conditions (1.2) and (1.3), in the norm corresponding to the scalar product

$$(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})_H = \int_a^b (u'^{(1)}u'^{(2)} + w'^{(1)}w'^{(2)} + \vartheta'^{(1)}\vartheta'^{(2)})\,\alpha_0 r_0\, d\xi$$

Space X_1 is a Hilbert space obtained by the closure of the set C_1 in the norm corresponding to the scalar product

$$(\mathbf{u}^{(1)}, \mathbf{u}^{(2)})_1 = \int_a^b (u^{(1)}u^{(2)} + w^{(1)}w^{(2)} + 12^{-1}\vartheta^{(1)}\vartheta^{(2)})\,\alpha_0 r_0\, d\xi$$

Let C_2 be a set of elements $\mathbf{u}(\xi, t)$ depending on the parameter t and such, that $\mathbf{u} \in C_1$, $\mathbf{u}_t \in X_1$ for any $-\infty < t < +\infty$, with finite norms

$$\max_t \|\mathbf{u}\|_1, \quad \max_t \|\mathbf{u}_t\|_1, \quad \int_0^\omega \|\mathbf{u}\|_H^2\, dt$$

The space $X_2(0, \omega)$ is the closure of the set C_2 in the norm corresponding to the scalar product

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$$(\mathbf{u}^{(1)}\cdot\mathbf{u}^{(2)})_{2,0,\omega}=\int_0^{\omega}[(\mathbf{u}_t^{(1)}\cdot\mathbf{u}_t^{(2)})_1+(\mathbf{u}^{(1)}\cdot\mathbf{u}^{(2)})_H]\,dt$$

The following lemmas are proved as in /5/.

Lemma 1. $H(a,b)$ represents the space $W=W_2^{0(1)}(a,b)\times W_2^{0(1)}(a,b)\times W_2^{0(1)}(a,b)$ and the norms of $H(a,b)$ and W are equivalent on $H(a,b)$.

Lemma 2. A complete system of vectors $\{\chi_m(\chi_{1m},\chi_{2m},\chi_{3m})\}$ exists in the space $H(a,b)$. The system can be regarded as orthogonal in $H(a,b)$, orthonormal in X_1 and such, that of $(\chi_{ip}\cdot\chi_{ip})_1=1$, then $(\chi_{jp}\cdot\chi_{jp})_1=0,\ i,\ j=1,2,3,\ p=1,\ldots,n,\ j\neq i$.

Lemma 3. $X_2(0,\omega)$ is a separable Hilbert space and a subset of elements of C_2, which can be represented in the form of finite sums $\Sigma d_k(t)\varphi_k$ (where $d_k(t)\in C^{(2)}(0,\omega)$ and satisfy (1.3) and $\varphi_k\in H(a,b)$) densely everywhere in it.

Lemma 4. The vector-function $\mathbf{u}_t$ regarded as an element of X_1 and $\mathbf{u}$ as an element of $H(a,b)$ are both functions of t, $0\leqslant t\leqslant\omega$, continuous almost everywhere.

3. Generalized solution and solvability of the problem.

Let the conditions

4) $$\mathbf{F}(t+\omega)=\mathbf{F}(t),\ \max_t\|\mathbf{F}\|_1<\infty,\ (\mathbf{F}=(F_1,F_2,F_3))$$

hold. Equations of motion of the shell can be written, according to the Hamilton—Ostrogradskii principle, in the form

$$\int_0^{\omega}\Big\{-(\mathbf{u}_t\cdot\delta\mathbf{u}_t)_1+\varepsilon(\mathbf{u}_t\cdot\delta\mathbf{u})_1+\int_a^b(T_1\delta\varepsilon_1+T_2\delta\varepsilon_2+Q\delta\gamma+M_1\delta\varkappa_1+M_2\delta\varkappa_2)\,\alpha_0r_0\,d\xi-(\mathbf{F}\cdot\delta\mathbf{u})_1\Big\}\,dt=0,\quad\delta\mathbf{u}=(\delta u,\delta w,\delta\vartheta)$$

where $\delta\mathbf{u}$ denotes a possible displacement.

Definition. Vector function $\mathbf{u}(\xi,t)$ satisfying the conditions that:

a) $\mathbf{u}(\xi,t+\omega)=\mathbf{u}(\xi,t),\ \mathbf{u}_t(\xi,t+\omega)=\mathbf{u}_t(\xi,t)$;

b) $\max_t\|\mathbf{u}_t\|_1,\ \max_t\|\mathbf{u}\|_H,\ \|\mathbf{u}\|_{2,0,\omega}$ are finite;

c) the Hamilton—Ostrogradskii equations hold for any $\delta\mathbf{u}\in H(a,b)$, strongly differentiable in t, shall be called the generalized, ω-periodic solution of the problem (1.1)—(1.3).

Using the accepted method of variational calculus, we can reduce the problem of obtaining a generalized, ω-periodic solution, to that of determining the solvability of the operator equation (1.1) in the space $X_2(0,\omega)$. Bubnov—Galerkin method can be used to obtain the generalized solution in approximate form. We construct a sequence $\{\mathbf{u}_n\}$ of the form $\mathbf{u}_n=q_1(t)\chi_1+\ldots+q_n(t)\chi_n$, where χ_m are defined in Lemma 2. The vector $(\mathbf{q}_n(t),\mathbf{q}_{nt}(t))=(q_1(t),\ldots,q_n(t),q_{1t}(t),\ldots,q_{nt}(t))$ is determined as a periodic solution of the following nonlinear system of ordinary differential equations:

$$(\mathbf{u}_{ntt}\cdot\boldsymbol{\chi}_m)_1+\varepsilon(\mathbf{u}_{nt}\cdot\boldsymbol{\chi}_m)_1+I_{nm}-(\mathbf{F}\cdot\boldsymbol{\chi}_m)_1=0 \tag{3.1}$$

$$I_{nm}=\int_a^b(T_{1n}\delta\varepsilon_{1m}+T_{2n}\delta\varepsilon_{2m}+Q_n\delta\gamma_m+M_{1n}\delta\varkappa_{1m}+M_{2n}\delta\varkappa_{2m})\,\alpha_0r_0\,d\xi,\quad(m=1,\ldots,n)$$

Here $T_{1n},\ldots,M_{2n}$ are obtained by replacing $\mathbf{u}$ by $\mathbf{u}_n$; the expressions $\delta\varepsilon_{1m},\ldots,\delta\varkappa_{2m}$, with the hypotheses of /3/ taken into account, have the form

$$\delta\varepsilon_{1m}=\alpha_0^{-1}\chi_{1m}'\cos\vartheta_n+\alpha_0^{-1}\chi_{2m}'\sin\vartheta_n,\quad\delta\varepsilon_{2m}=r_0^{-1}\chi_{1m},\quad\delta\varkappa_{1m}=\alpha_0^{-1}\chi_{3m}'$$
$$\delta\gamma_m=\alpha_0^{-1}\chi_{2m}'\cos\vartheta_n-\alpha_0^{-1}\chi_{1m}'\sin\vartheta_n-\chi_{3m},\quad\delta\varkappa_{2m}=-r_0^{-1}\chi_{3m}\cos\vartheta_n$$

Theorem. Let the conditions 1)—4) hold, and let $\{\chi_m\}$ be a system of vector functions defined in Lemma 2. Then

a) system of equations (3.1) has at least one ω-periodic solution for any value of n;

b) the set of approximations $\{\mathbf{u}_n\}$ is weakly compact in $X_2(0,\omega)$;

c) every weak limit of $\{\mathbf{u}_n\}$ in $X_2(0,\omega)$ represents a generalized, ω-periodic solution of the problem (1.1)—(1.3).

The proof of the theorem is centered on confirming the dissipative character /6/ of the equations (3.1). The equations of the Bubnov—Galerkin method in the theory of nonshallow shells of revolution differ from the corresponding equations of the theory of thin plates /1/ and shallow shells /2/ in the following aspects. Let the following positive-definite functional of potential energy of the shell be given on the space $H(a,b)$:

$$\Phi_n=\Phi(\mathbf{u}_n)=\frac{1}{2}\int_a^b(T_1\varepsilon_1+T_2\varepsilon_2+Q\gamma+M_1\varkappa_1+M_2\varkappa_2)\,\alpha_0r_0\,d\xi$$

In the theory of plates, the form Φ can be written in terms of $q_m(t)$ as a sum $\Phi_n=\Phi_{2n}+\Phi_{4n}$ of the forms of second and fourth degree. In the theory of shallow shells we have

$\Phi_n = \Phi_{2n} + \Phi_{3n} + \Phi_{4n}$, where Φ_{4n} is a third degree functional in $q_m(t)$. In the theory of non-shallow shells of revolution the functional Φ_n is no longer a sum of homogeneous functionals.

To prove the theorem we multiply the equations (3.1) by $q_{mt}(t)$, sum over m from 1 to n, and add the resulting expressions

$$\frac{d}{dt}(2^{-1}\|\mathbf{u}_{nt}\|_1^2 + \Phi_n) = (\mathbf{F}\cdot\mathbf{u}_{nt})_1 - \varepsilon\|\mathbf{u}_{nt}\|_1^2$$

Next we introduce the function

$$V_n(t) = V(\mathbf{q}_n(t), \mathbf{q}_{nt}(t)) = 2^{-1}\|\mathbf{u}_{nt}\|_1^2 + \Phi_n + \alpha\sum_{m=1}^{n}(\mathbf{u}_{nt}\cdot\chi_m)_1(\mathbf{u}_n\cdot\chi_m)_1 + \beta\sum_{m=1}^{n}(\mathbf{u}_n\cdot\chi_m)_1^2$$

and impose the following constraints on the constants $\alpha > 0$ and $\beta > 0$:

$$2^{-1} - \alpha\varepsilon_1^2 > 0, \quad \beta - 2^{-1}\alpha\varepsilon_1^{-2} > 0$$

Taking into account the Young's inequality with constant ε_1^2 we can show the sufficiency of the above inequalities for the positive definiteness of $V_n(t)$. Using (3.1), we obtain the following expression for the derivative $V_{nt}(t)$:

$$V_{nt}(t) = (\mathbf{F}\cdot\mathbf{u}_{nt})_1 - \varepsilon\|\mathbf{u}_{nt}\|_1^2 + 2\beta\sum_{m=1}^{n}(\mathbf{u}_{nt}\cdot\chi_m)_1(\mathbf{u}_n\cdot\chi_m)_1 + \alpha\sum_{m=1}^{n}(\mathbf{u}_{nt}\cdot\chi_m)_1^2 +$$

$$\alpha\sum_{m=1}^{n}(\mathbf{u}_n\cdot\chi_m)\{-\varepsilon(\mathbf{u}_{nt}\cdot\chi_m)_1 - I_{nm} + (\mathbf{F}\cdot\chi_m)_1\}$$

Let $\alpha\varepsilon = 2\beta$. The Young's inequalities with ε_2^2 and ε_3^2 yield

$$V_{nt}(t) \leqslant -a\|\mathbf{u}_{nt}\|_1^2 + b\|\mathbf{F}\|_1^2 - \alpha\Phi_n^\circ$$

$$\Phi_n^\circ = \Phi_n^\circ(t) = \sum_{m=1}^{n}(\mathbf{u}_n\cdot\chi_m)_1 I_{nm} - 2^{-1}\varepsilon_3^2\|\mathbf{u}_n\|_1^2$$

Let $a = \varepsilon - 2^{-1}\varepsilon_2^2 - 2\alpha > 0$, $b = 2^{-1}\varepsilon_2^{-2} + 2^{-1}\varepsilon_3^{-2}$; and let $S(1, 0)$ be a sphere of unit radius in the space $H(a, b)$ with its center at the zero: $\|\mathbf{u}\|_H = 1$. Projecting the sphere $S(1, 0)$ with help of the mapping $u = R^2u_1, w = R^2w_1, \vartheta = R\vartheta_1$ where $R > 0$ is a constant, we find the ellipsoid $C(R, 0)$ in the space $H(a, b)$. When the constant $R > 1$ is fixed, the ellipsoid becomes a boundary of a connected convex region containing a unit sphere with center at the zero of the space $H(a, b)$.

Lemma 5. Let $C(R, 0)$ be an ellipsoid belonging to the space $H(a, b)$, of sufficiently large radius R, independent of t. If an element $\mathbf{u}_n(t)$ belonging to the space $H(a, b)$ arrives, for every fixed $-\infty < t < +\infty$ and all n, at some value $t = t^*$ at the ellipsoid $C(R, 0)$ of sufficiently large radius, then the following inequality holds:

$$\Phi_n^\circ(t^*) \geqslant \delta_4R^4 - \delta_3R^3 - \delta_2R^2 - \delta_1R - \delta_0 \tag{3.2}$$

where $\delta_0, \ldots, \delta_4$ are constants independent of $\mathbf{u}_n(t^*)$.

To prove Lemma 5, we assume that the positive definiteness of the form

$$\Psi_n = B(\varepsilon_{1n}^2 + \varepsilon_{2n}^2 + 2\nu\varepsilon_{1n}\varepsilon_{2n}) + C\gamma_n^2 + D(\varkappa_{1n}^2 + \varkappa_{2n}^2 + 2\nu\varkappa_{1n}\varkappa_{2n})$$

implies the positive definiteness of the form

$$2m_3(\varepsilon_{1n}^2 + \varepsilon_{2n}^2 + \gamma_n^2 + \varkappa_{1n}^2 + \varkappa_{2n}^2) \leqslant \Psi_n$$

Here and henceforth $m_i > 0$ are constants independent of n. Let us write the inequalities

$$\Phi_n \geqslant m_3\int_a^b(\varepsilon_{1n}^2 + \varepsilon_{2n}^2 + \gamma_n^2 + \varkappa_{1n}^2 + \varkappa_{2n}^2)\alpha_0r_0\,d\xi \geqslant m_4(J_{1n} - J_{2n}) \tag{3.3}$$

$$J_{1n} = \int_a^b[\alpha_0^{-2}(u_n'^2 + w_n'^2 + \vartheta_n'^2) + r_0^{-2}u_n^2]\alpha_0r_0d\xi$$

$$J_{2n} = \int_a^b[3\alpha_0^{-1}(|u_n'| + |w_n'|) + 2\alpha_0^{-1}|\vartheta_0'\vartheta_n'|]\alpha_0r_0\,d\varepsilon$$

From the properties of the space $H(a, b)$ and conditions 1) and 2), it follows that

$$m_5\|\mathbf{u}_n\|_H^2 \leqslant J_{1n} \leqslant m_6\|\mathbf{u}_n\|_H^2, \quad J_{2n} \leqslant m_7\|\mathbf{u}_n\|_H \tag{3.4}$$

The functional Φ_n° is transformed thus

$$\Phi_n^\circ = \int_a^b\{T_{1n}[\varepsilon_{1n} + 1 - \cos(\vartheta_n - \vartheta_0)] + T_{2n}\varepsilon_{2n} + Q_n[\gamma_n + \sin(\vartheta_n - \vartheta_0) -$$

$$\vartheta_n] + M_{1n}(\varkappa_{1n} - \alpha_0^{-1}\vartheta_0') + M_{2n}(-r_0^{-1}\vartheta_n\cos\vartheta_n)\}\alpha_0r_0\,d\xi - 2^{-1}\varepsilon_3^2\|\mathbf{u}_n\|_1^2$$

and this yields, with the help of elementary inequalities,

$$\Phi_n^\circ \geqslant 2\Phi_n - 2^{-1}\varepsilon_3^2\|\mathbf{u}_n\|_1^2 - \int_a^b[2|T_{1n}| + |Q_n|(1 + |\vartheta_n|) + |M_{1n}||\alpha_0^{-1}\vartheta_0'| + |M_{2n}||r_0^{-1}|(2 + |\vartheta_n|)]\alpha_0r_0\,d\xi \tag{3.5}$$

Theorems of imbedding the space $H(a, b)$ in the Hölder space $H^\alpha(a, b)$ for $\alpha < 2^{-1}$ /7/ and the inequalities (3.3)—(3.5) together yield the inequality

$$\Phi_n^\circ \geqslant m_8 \|\mathbf{u}_n\|_H^2 - m_9 \|\mathbf{u}_n\|_H - 2^{-1}\varepsilon_3^2 \|\mathbf{u}_n\|_1^2 - \int_a^b [2\,|T_{1n}| + |Q_n|\,(1 + |\vartheta_n|) +$$
$$|M_{1n}|\,|\alpha_0^{-1}\vartheta_0'| + |M_{2n}|\,|r_0^{-1}|\,(2 + |\vartheta_n|)]\,\alpha_0 r_0\,d\xi$$

Choosing ε_3^2 so that $m_8 - 2^{-1}\varepsilon_3^2 \geqslant m_{10} > 0$ (which is always possible), we can obtain, on the ellipsoid $C(R, 0)$, the estimate (3.2) sought. This proves Lemma 5. Further arguments needed to prove the theorem follow those given in /2/.

Note. If e.g. we choose the constants $\varepsilon_1^2, \varepsilon_2^2, \alpha, \beta,$ corresponding to the inequalities

$$\alpha < 4^{-1}\varepsilon,\ \varepsilon^{-1} < \varepsilon_1^2 < 4\varepsilon^{-1},\ \beta < 8^{-1}\varepsilon^2,\ \varepsilon_2^2 < \varepsilon,\ \alpha\varepsilon = 2\beta$$

then all restrictions imposed on them will hold.

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